

The Missing Star: Measuring the Natural Rate of Exchange

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October 26, 2026

Abstract

This paper contributes quarterly time-varying estimates of the natural rate of exchange (q_t^*) for the United States from 1970Q1 to 2025Q4, defined as the real effective exchange rate consistent with output at potential, stable inflation, and a stable trade balance. A semi-structural open-economy state-space model captures the role of the real exchange rate in macroeconomic dynamics and is estimated by a multi-stage maximum likelihood procedure. Misalignments between the real effective exchange rate and the time-varying natural rate of exchange are sizeable and persistent across the historical sample, with filtered (one-sided) estimates closely tracking their smoothed (two-sided) counterparts.

JEL: F31, F41, C32, E32

Keywords: natural rate of exchange, equilibrium real exchange rate, state-space models

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1 Introduction

The real exchange rate is among the central macroeconomic quantities for which no standard measure of equilibrium is in routine use. In contrast, well established measures of potential output and the natural rates of unemployment and interest are widely used in macroeconomic analysis, with estimates reported alongside measures of uncertainty and revised as new data is accumulated. Comparable time-varying estimates of the real exchange rate consistent with internal and external stability are less common within the existing literature, which focuses largely on estimating the level of equilibrium rather than its evolution over time.

The absence of a time-varying estimate of the equilibrium real exchange rate matters because claims of misalignment cannot otherwise be meaningfully evaluated. If the natural rate changes materially over time, an estimate of its level alone provides an unreliable reference point for judging the pressures the real exchange rate places on the economy. Treating the equilibrium as a latent state within a semi-structural model allows its path to be estimated jointly with the relationships that define it. Movements in the real exchange rate can then be decomposed into shifts in the equilibrium and departures from it, allowing the size and persistence of misalignment to be quantified.

Existing estimates of the equilibrium real exchange rate fall into three broad categories, namely purchasing power parity, the behavioural equilibrium exchange rate and the macroeconomic balance approach (Ca' Zorzi et al., 2022). In the theory of purchasing power parity, the equilibrium real exchange rate is assumed constant, measured by the long-run mean of the real exchange rate, deviations from which are persistent (Rogoff, 1996; Taylor and Taylor, 2004). Behavioural equilibrium exchange rate approaches allow for variation, regressing the real exchange rate on fundamentals such as net foreign assets, the terms of trade and relative productivity (Clark and MacDonald, 1999; Lane and Milesi-Ferretti, 2004; Ricci et al., 2013). The macroeconomic balance approach, defines the equilibrium as the real rate consistent with output at potential and a current account at a norm estimated from medium-term fundamentals (Chinn and Prasad, 2003).

In the latter two approaches misalignment is understood as the part of the real exchange rate that observed fundamentals leave unexplained. Such fundamentals, however, account for only part of the movement in the exchange rate (Meese and Rogoff, 1983; Cheung et al., 2005; Itskhoki and Mukhin, 2021), which has also been linked to financial forces, such as risk premia and the liquidity value of government bonds, that are difficult to measure over long samples (Gabaix and Maggiori, 2015; Engel, 2016). In contrast, pure statistical time-series decompositions sit at the other extreme, separating permanent from transitory movements without the imposition of any macroeconomic structure. More recent unobserved-components models

([Holston et al., 2017](#)) occupy a desirable middle ground, extracting the equilibrium from the joint behaviour of macroeconomic variables; retaining a degree of theoretical structure while allowing the equilibrium to evolve as a latent trend, so that it is neither fixed nor confined to the observed fundamentals.

This paper applies a decomposition of this kind to the real exchange rate. The natural rate of exchange q_t^* is defined as the real exchange rate consistent with output at potential, stable inflation, and a stable external position. In this paper, q_t^* is estimated for the United States from 1970Q1 to 2025Q4 on a quarterly basis, using a semi-structural open-economy model in four observables, namely output, inflation, the real effective exchange rate and net exports. The real exchange rate gap operates through three channels, affecting demand in the dynamic IS curve, prices through pass-through in the Phillips curve and finally net exports through expenditure switching in the trade balance. The division between the natural rate and the gap rests on a median-unbiased estimate of the natural rate’s volatility. The natural rate combines a latent trend with an observed component in the terms of trade, reflecting the close link between commodity prices and real exchange rates, including that of the dollar ([Chen and Rogoff, 2003](#); [Ayres et al., 2020](#)).

Three core findings emerge from this analysis. Misalignment is substantial, reaching 18.0 percent at the dollar peak of the mid-1980s and 22.3 in the opposite direction during the late 1970s. It is highly persistent, maintaining the same sign for 4 years on average. The natural rate itself falls by roughly 33 percent to a low in 2008Q3 and then rises by 19 percent, such that relative purchasing power parity provides a poor description of the data at this horizon. In the model, the observed terms of trade component accounts for 55 percent of its quarterly variation and a latent trend for the rest. Estimates formed using information through each time period are closely aligned with those formed on the full sample, the observed component largely unrevised as later data accumulates.

The remainder of this paper is structured as follows. Section 2 sets out the semi-structural model, Section 3 its estimation, Section 4 the results and Section 5 their robustness; Sections A and B document the data and the full state-space specification.

2 Framework

2.1 Theoretical background

The equations estimated below are reduced forms of the canonical open-economy New Keynesian structure, together with a standard model of trade volumes in which home and foreign goods are imperfect substitutes.

In the open-economy models of [Galí and Monacelli \(2005\)](#) and [Clarida et al. \(2002\)](#), log-linearisation about a symmetric steady state yields three key relations: an aggregate demand equation in which the output gap depends on its expected future value, the real interest rate gap and the real exchange rate, the last through expenditure switching between home and foreign goods; the Phillips curve in which inflation is contingent on expected inflation, the output gap and the real exchange rate, and a risk-adjusted interest parity condition relating the expected movement in the real exchange rate to the real interest differential.

Replacing expected future values with their lagged counterparts, as is standard in semi-structural measurement models of this class, yields backward-looking approximations whose coefficients are convolutions of the underlying structural parameters. The coefficients that are estimated in this paper are therefore reduced form throughout, and we attach no structural interpretation to them beyond their signs. In the trade relation, import and export volumes depend on relative prices and on activity at home and abroad; quantities respond to prices only gradually, which is why the real exchange rate enters with a lag.

We depart from these general structures in two respects. Trend output and the natural rate of exchange are treated as stochastic processes rather than assumed constant, following the unobserved-components tradition and the natural rate of exchange is given an observed component in the terms of trade, so that part of the equilibrium rate is tied to a measured fundamental as in behavioural equilibrium models. The terms of trade are the fundamental driving force for which a long quarterly series exists; an improvement in the relative price of exports raises the real exchange rate consistent with any given external position. Net foreign assets would be the natural complement ([Lane and Milesi-Ferretti, 2007](#)), but quarterly data for the United States is generally scarce beginning only in 2006.

2.2 Measurement equations

Four series are observed: y_t , the log of real output; π_t , core inflation; q_t , the log real effective exchange rate, rising with an appreciation; and n_t , net exports as a share of nominal output. Three further series enter as observed exogenous variables: the terms of trade τ_t , the real rate gap $\tilde{r}_t$ between the ex-ante real rate and the natural rate of interest, and the nominal long-rate differential Δi_t between the United States and a GDP-weighted average of the G7 economies (refer to Section A for sources and construction).

The law of motion for the natural rate of exchange is therefore composed of a terms of trade component and an unobserved component as follows:

$$q_t^* = \beta_\tau \tau_t + \mu_t, \tag{1}$$

The measurement equations of the system are presented as follows:

$$y_t = y_t^* + a_1 \tilde{y}_{t-1} + a_2 \tilde{y}_{t-2} + a_q \tilde{q}_{t-1} + a_r \tilde{r}_{t-1} + \varepsilon_t^y, \quad (2)$$

$$\pi_t = b_\pi \pi_{t-1} + (1 - b_\pi) \bar{\pi}_{t-2,t-4} + b_y \tilde{y}_{t-1} + b_q \tilde{q}_{t-1} + \varepsilon_t^\pi, \quad (3)$$

$$q_t = q_t^* + \rho_{q,1} \tilde{q}_{t-1} + \rho_{q,2} \tilde{q}_{t-2} + c_y \tilde{y}_{t-1} + c_r \Delta i_{t-1} + \varepsilon_t^q, \quad (4)$$

$$n_t = n_{t-1} + \gamma_q \tilde{q}_{t-2} + \gamma_y \tilde{y}_t + \gamma_\tau \Delta \tau_t + \varepsilon_t^n. \quad (5)$$

where $\tilde{y}_t = y_t - y_t^*$, and $\tilde{q}_t = q_t - q_t^*$. Equation (2) is the IS curve, written as an equation for output; a_q and a_r are the responses of demand to the real exchange rate gap and to the real rate gap. The output gap enters with two lags, as selected by both the Akaike and Schwarz information criteria. The real rate enters the dynamic IS curve in deviation from its natural rate, sourced directly from [Holston et al. \(2023\)](#).

Equation (3) is an accelerationist Phillips curve in which the coefficients on lagged inflation sum to unity, where $\bar{\pi}_{t-2,t-4}$ represents the average of the second through fourth lags of inflation. This equation has neither an intercept nor a term in the level of inflation, and thus carries no implication about the long-run inflation rate, which is stable only when both the output gap and the exchange rate gap are zero.

Equation (4) is the interest-parity relation, written as an equation for the real exchange rate; $\rho_{q,1}$ and $\rho_{q,2}$ govern the speed at which it returns to the natural rate and c_r its response to the interest differential. Modelling q_t is what makes $\tilde{q}_t$ a mean-reverting gap rather than an unrestricted residual. With Δi and $\tilde{y}$ constant the gap settles at $(c_r \Delta i + c_y \tilde{y}) / (1 - \rho_{q,1} - \rho_{q,2})$, such that the natural rate is the rate consistent with internal and external balance at a zero long-run rate differential.

Equation (5) outlines the law of motion for the natural rate of exchange q_t^* , defined as the real exchange rate at which net exports are neither improving nor deteriorating. This should be read as a condition on the change in the external position, rather than its level, so that the natural rate is not pinned to any particular trade balance and carries no implication drawn from an intertemporal budget constraint nor from a target for net foreign assets. The natural rate of exchange estimated in this paper is accordingly the rate at which net exports are free of exchange rate and cyclical pressure; a weaker condition than the external sustainability requirement often found in fundamental equilibrium exchange rate approaches.

Net exports (5) is specified in first differences, which the data favour over the specification in levels. Augmented Dickey Fuller (ADF) tests fail to reject a unit root in net exports, with or without a trend, KPSS tests reject stationarity around both, and neither the lagged level nor a drift is significant when added to the differenced equation. We also exclude a latent

component from (5), since any component flexible enough to track net exports would also absorb the exchange rate variation the equation is meant to identify. Section 5 reports results for the specification in levels.

As net exports are measured in nominal terms, (5) also includes the change in the terms of trade. A shift in relative prices changes the nominal trade balance before volumes have time to adjust, and without this term the valuation effect would be attributable to the exchange rate gap. The real exchange rate appears in the equation only through the gap $\tilde{q}_{t-2}$. Given movements in q_t^* are by construction consistent with external balance, they do not exert any pressure on net exports, which therefore respond only to deviations of the real exchange rate from its equilibrium.

2.3 Transition equations

Trend output is assumed to evolve as a random walk with drift given by trend growth, which in turn follows a random walk, whereas the latent component of the natural rate is modelled as a driftless random walk, consistent with Laubach and Williams (2003):

$$y_t^* = y_{t-1}^* + g_{t-1} + \varepsilon_t^{y^*}, \quad (6)$$

$$g_t = g_{t-1} + \varepsilon_t^g, \quad (7)$$

$$\mu_t = \mu_{t-1} + \varepsilon_t^\mu. \quad (8)$$

All disturbances are Gaussian, mutually independent and serially uncorrelated. Stationarity of the autoregressive terms in equations (2) and (4) implies both gaps are mean-reverting by construction and (6)-(8) carry the permanent components of output and the exchange rate.

2.4 State-space form and the pandemic

The model (2)-(8) is a linear Gaussian state-space system in seven states:

$$\xi_t = (y_t^*, y_{t-1}^*, y_{t-2}^*, g_t, \mu_t, \mu_{t-1}, \mu_{t-2})'. \quad (9)$$

As the output gap and the real exchange rate gap are each an observed series minus a trend, the state vector need only contain the trends and the lags required to form them. The state vector therefore includes two lags of y_t^* and μ_t , as both gaps enter the model up to $t - 2$. Each measurement equation is consequently linear in ξ_t , with the observed terms collected in a time-varying intercept, and each equation carrying its own disturbance term. Section B presents the complete state-space representation at each stage of the estimation.

Shock standard deviations are scaled over the pandemic following [Lenza and Primiceri \(2022\)](#): the four measurement disturbances are multiplied by κ_0 in 2020Q2, κ_1 in 2020Q3, and thereafter by $1 + (\kappa_1 - 1)\rho_{\kappa}^j$, with all three parameters estimated. The trend disturbances in (6)-(8) are left unscaled, so that the contraction and recovery of 2020 are attributed to the gaps rather than to trend output and the natural rate. Section 5 reports three alternatives, including treating 2020Q2-Q4 as missing.

3 Estimation

The state-space system is estimated by maximum likelihood over 1970Q1-2025Q4, with the likelihood being evaluated by the Kalman filter. Trend components of the model are given a wide prior centred on a regression on the first ten years of data, and the first four quarters are excluded from the likelihood function for initialisation.

Two features of the parameter space require attention. Both gaps must be stationary for any decomposition to be meaningful and are therefore parameterised through their partial autocorrelations, making (2) and (4) stationary by construction; (5) has a unit coefficient on n_{t-1} and no intercept by construction. The response of net exports to the gap is restricted by expenditure switching, such that $\gamma_q \leq 0$. In the absence of this restriction the likelihood has a second mode, roughly eight log points higher, in which a large positive γ_q lets a small, transitory gap fit the noise in net exports with minimal misalignment. The likelihood is also multimodal in the trend/cycle division, so estimation starts from a grid of initial values, and every alternative specification also from the baseline estimates.

Estimation follows the three-stage procedure of [Laubach and Williams \(2003\)](#) in which the median-unbiased estimator is applied in stages to avoid the pile-up problem ([Stock and Watson, 1998](#)). Stage I first imposes constant trend growth, $\sigma_g = 0$, and recovers the ratio $\lambda_g = \sigma_g/\sigma_{y^*}$ from the median-unbiased estimator applied to the smoothed path of the trend growth rate. Stage II imposes λ_g , holds the latent component of the natural rate constant, and recovers $\lambda_q = \sigma_{\mu}/\sigma_q$ from a test for a shift in the intercept of (4). Stage III then imposes both ratios and estimates the remaining parameters.

We report both one-sided $E[\xi_t \mid \mathcal{I}_t]$, and two-sided $E[\xi_t \mid \mathcal{I}_T]$ estimates, where $\mathcal{I}_t$ is the information set at t . The reported intervals combine filter uncertainty about the states with uncertainty about the parameters. The latter is propagated by drawing from the asymptotic distribution of the estimates in the transformed space and re-running the filter and smoother for each draw ([Hamilton, 1986](#)). Uncertainty about the imposed ratio λ_q is not included, so the intervals are conditional on it.

3.1 Identification of the trend/cycle division

Two shock variances, those of trend growth and of the latent component of the natural rate, are only weakly determined by the likelihood, which is a well-known feature of unobserved-components models in such applications (Laubach and Williams, 2003). For the trend growth rate, the ratio $\lambda_g = \sigma_g/\sigma_{y^*}$ is estimated in Stage I by the median-unbiased estimator of Stock and Watson (1998) and then held fixed. The test underlying that stage is informative, and the resulting ratio, $\lambda_g = 0.0684$, is reported in Table 2.

The variance of the natural rate requires a separate restriction. Net of the terms-of-trade component, the real exchange rate is the sum of the latent trend and the gap by construction, so that with $\varphi_t \equiv q_t - \beta_\tau \tau_t = \mu_t + \tilde{q}_t$,

$$\text{var}(\Delta\varphi_t) \approx \sigma_\mu^2 + \text{var}(\Delta\tilde{q}_t), \quad (10)$$

where $\text{var}(\Delta\tilde{q}_t)$ is driven mainly by the gap's own disturbances and so is of the order of σ_q^2 . The likelihood pins down the sum sharply. How it divides between the two terms depends on the dynamics of the gap and on how the gap enters the other equations. With two lags in (4) and the valuation term in (5), the likelihood is informative about this division. Leaving both ratios free, the profile likelihood over σ_μ peaks at roughly 0.75, and a natural rate that absorbs almost all of the variance in (10) is rejected.

The likelihood therefore leaves the volatility of the natural rate of exchange largely undetermined, and it is instead fixed through the ratio $\lambda_q = \sigma_\mu/\sigma_q$, which governs how much of (10) is permanent. This ratio is estimated in Stage II. With the latent component of q^* held constant, (4) implies a constant intercept, and the exponential Wald test of Andrews and Ploberger (1994) for a shift in that intercept at an unknown date is then mapped into a median-unbiased estimate of $\lambda_q = 0.39$.

Taken alone, the test has insufficient power to discriminate between values of λ_q . Since the innovation to μ also enters the unobserved component in (4), the signal-to-noise ratio available to the test cannot exceed $1 - \rho_{q,1} - \rho_{q,2} \approx 0.090$, whatever the true value of λ_q , and its 90 percent interval accordingly spans the whole simulated grid.

Despite these differences, the two procedures arrive at much the same answer. Estimating λ_q jointly with the other parameters by maximum likelihood gives 0.38 (Panel B of Table 3), almost identical to the Stage II estimate, so imposing the latter costs essentially nothing in fit. Towards the upper end of the Stage II interval the fit deteriorates noticeably, so although the test alone cannot bound λ_q , the likelihood does rule out a highly volatile natural rate. The imposed ratio is therefore consistent with both procedures, and the data are informative about how much the natural rate can move.

4 Results

4.1 Parameters

Estimates are reported in Table 2. The real exchange rate gap is persistent but stationary, with estimates of $\rho_{q,1}$ and $\rho_{q,2}$ implying a half-life of roughly two years for a disturbance to the gap. Misalignment itself also inherits the persistence of the interest rate differential (Section 4.2), so this is not directly comparable with the three-to-five year range for deviations from purchasing power parity often cited in the literature (Rogoff, 1996).

The open-economy relations within the model are estimated precisely and with the signs that are predicted by theory. An improvement in the terms of trade appreciates the natural rate, with a coefficient of 0.588; Section 5 shows this is identified from co-movement within the quarter. A widening of the differential between United States and foreign long-term yields appreciates the real exchange rate relative to its natural rate, the interest-parity channel in (4); the implied long-run effect, $c_r/(1 - \rho_{q,1} - \rho_{q,2}) = 7.3$ log points per percentage point. The real rate gap enters the IS curve negatively, and an improvement in the terms of trade raises nominal net exports on impact ($\gamma_\tau = 0.081$, $t = 5.9$).

The real exchange rate gap has only small and imprecisely estimated effects on demand and prices, as might be expected given the modest role of trade in the United States economy. The positive estimate of a_q arguably reflects the tendency of the dollar to appreciate when domestic demand is strong. In net exports, where expenditure switching should be clearly visible, the effect has the expected negative sign and is significant once valuation effects are accounted for. As the effects on output and inflation are small, the estimated gap is largely insensitive to them. Estimates of misalignment therefore do not hinge on coefficients the data determine only loosely, with the exchange rate equation providing most of the information about the split between the natural rate and the gap.

The remaining parameter estimates are largely conventional. Trend growth of potential output declines from 3.2 percent to 2.2, and the slope coefficient of the Phillips curve is small. These results are consistent with closed-economy estimates of this kind, including those of Holston et al. (2017), suggesting that the open-economy extension leaves the domestic block of the state-space model largely intact.

4.2 The natural rate of exchange

Figure 1 plots two-sided estimate against the actual real exchange rate, and Table 1 reports both estimates at selected dates. The natural rate of exchange declines steeply through the 1970s and once more in the late 1980s, drifts within a narrow range until its low in 2008Q3,

and then increases by 17 percent towards the end of the sample. In the model, the observed terms of trade accounts for 55 percent of the variance of the quarterly change in q_t^* , with the latent trend accounting for the rest. In levels the two components vary by similar amounts, with standard deviations of 5.6 and 4.4 percent respectively. The observed component peaks prior to the first oil shock and reaches its low during the commodity boom.

Deviations of the actual exchange rate from the natural rate of exchange are persistent. Periods of misalignment hold the same sign for 4 years on average (16 quarters), so the dollar experiences long stretches on one side of its natural rate. Since (4) is linear, the gap can be divided into three components; one driven by the differential, another by the output gap, and accumulated disturbances. The differential accounts for 64 percent of the variance of the gap, whereas the output gap contribution is negligible. Excluding the differential, misalignment keeps its sign for only 8 quarters on average. A substantial part of the measured misalignment is therefore the dollar moving in line with interest rate parity.

Figure 2 plots the time-varying gap between the actual and natural rates of exchange. Strong periods of misalignment coincide with well-known historical swings in the dollar. For instance, the dollar is well below its natural rate through the late 1970s, with the gap reaching about 20 percent in 1978Q4, during the dollar crisis. It then rises sharply to about 20 percent above the natural rate in early 1985, shortly before the Plaza Accord. The dollar is again undervalued in the mid-1990s, by around 12.2 percent, before the shift to a strong-dollar policy. By 2002 it is 11.4 percent above its natural rate, at a time when the sustainability of the United States external position was widely debated (Obstfeld and Rogoff, 2005). It is overvalued again through the 2020s and ends the sample 8.2 percent above its natural rate.

4.3 Filtered and smoothed estimates

Table 1 reports one-sided alongside two-sided estimates. The difference between them reveals how much the estimate for a given quarter is revised once later data become available. These revisions are small relative to the gap and concentrated in the first few quarters, where the filter has little history to draw on, and around turning points. The 1985 peak is the clearest example outside the opening quarters. Given information available at that time, the dollar already appears well above its natural rate, albeit the size of the overvaluation is understated by more than a third. The terms-of-trade component contributes to this stability, since only the latent trend is revised, and the estimates are noticeably less stable without it. Such one-sided estimates employ current-vintage data and full-sample parameters, so they capture the end-of-sample problem but not the errors from data revisions and re-estimation, which are found to be substantial, particularly for the output gap (Orphanides and van Norden (2002)).

5 Robustness and sensitivity

Table 3 reports a range of alternative specifications. Treating the coronavirus pandemic period as missing, or dropping the volatility correction altogether, leaves the path of misalignment almost unchanged, but moves the end-of-sample gap to 5.5 and 9.7 percent respectively. This is partly because a single scale factor applies to all four measurement equations, so the scaling of the exchange rate equation in 2020 is driven largely by output. Dropping either the terms of trade or the interest differential worsens the fit significantly, while writing (5) in levels makes little difference. Estimates on a sample ending in 2019Q4, with λ_g re-estimated and λ_q held at its full-sample value, are also very close to the baseline.

Four further checks concern the components of the model. The first relates to the terms of trade. Since they are defined as a ratio of dollar deflators, the dollar itself may influence them, and β_τ is identified mainly from within-quarter co-movement. However, entering the terms of trade component with a lag removes that channel. The coefficient then falls to essentially zero and the fit worsens markedly, although the path of misalignment changes much less. The terms-of-trade component is therefore best read as a within-quarter association rather than the effect of a fundamental. The second check concerns the interest differential. Measuring it in real terms, by subtracting each country's own inflation from its yield, reduces both the response of the exchange rate to the differential and the size of the gap, so the interest-parity part of misalignment depends on how the differential is measured.

The remaining two checks concern features that distinguish the baseline estimate from a more simpler specification, namely the second lag in equation (4) and the valuation term in equation (5). Both are supported by the data. With a single lag in the former the fit is worse and the residuals are autocorrelated, which would thus invalidate the white-noise assumption underlying the Stage II test. Dropping the valuation term from the latter also worsens the fit considerably. Removing both, however, leaves the path of misalignment almost unchanged. Their contribution lies instead in how precisely the data determine the wedge between the natural rate of exchange and the gap.

Across these alternatives, past episodes of misalignment are estimated more robustly than more recent ones. The size of the 1985 overvaluation is similar in almost every specification, and falls appreciably only when the interest differential is removed or measured in real terms. The gap at the end of the sample is less stable, disappearing when the differential is dropped or deflated by recent inflation, and widening when the terms of trade are excluded. Much of the dollar's recent overvaluation therefore reflects the interest differential, while the terms of trade attribute part of its recent strength to a rise in the natural rate itself. Persistence of the gap and the volatility of the natural rate, by contrast, change little across specifications.

6 Conclusion

This paper estimates a natural rate of exchange for the United States at quarterly frequency from 1970Q1 to 2025Q4, with sampling uncertainty and one-sided as well as two-sided estimates, by extending an unobserved-components framework to an open economy and identifying the trend/cycle division by way of median-unbiased estimation. The resulting series assigns magnitudes and intervals to principal episodes of the dollar's movements from the 1970s, and changes little when information through each date is used.

Baseline estimates show that the natural rate of exchange moves considerably over time, falling for much of the sample before rising after the financial crisis of 2008, with the terms of trade accounting for a large share of its short-run variation. Departures of the dollar from its natural rate are both large and long-lived, and much of their size and persistence can be traced to the longer-term interest differential. Estimates using information available at each date are close to those based on the full sample, and the historical episodes of misalignment are robust across a range of specifications.

These results suggest that judgements about dollar misalignment are heavily dependent on allowing the equilibrium to vary over time. An invariant benchmark attributes the secular decline in the real exchange rate to misalignment, whereas a purely statistical trend absorbs much of what is better seen as persistent departures from equilibrium. Treating the natural rate as a latent state, disciplined by the terms of trade and by the relationships that define it, places the estimates between these two extremes.

Two extensions follow naturally from this research. The first is to incorporate net foreign assets to the natural rate, which requires interpolating a long annual series and would discipline the level of the equilibrium rather than its variance alone. The second is to estimate the system jointly with a natural rate of interest, since the interest-parity relation links the two equilibria and the information in the exchange rate ought to sharpen estimates of both.

Table 1: The natural rate of exchange and dollar misalignment

Quarter	q	Natural rate q^*				Misalignment $q - q^*$			
		One-sided		Two-sided		One-sided		Two-sided	
		Est.	90% band	Est.	90% band	Est.	90% band	Est.	90% band
1970Q1	474.7	482.5	[466.0, 500.0]	477.9	[467.9, 485.8]	-7.9	[-25.3, 8.7]	-3.2	[-11.1, 6.7]
1979Q4	443.4	454.8	[444.7, 463.6]	453.6	[446.1, 459.1]	-11.4	[-20.2, -1.4]	-10.2	[-15.7, -2.8]
1985Q1	474.5	463.3	[455.3, 470.8]	456.5	[450.6, 462.0]	11.2	[3.7, 19.2]	18.0	[12.5, 23.9]
1990Q1	440.6	447.6	[440.6, 455.4]	446.1	[440.1, 451.7]	-7.0	[-14.8, -0.0]	-5.5	[-11.1, 0.4]
1995Q2	431.5	442.8	[435.5, 451.0]	443.6	[437.7, 449.3]	-11.4	[-19.6, -4.1]	-12.2	[-17.9, -6.2]
2002Q1	459.0	451.1	[443.9, 458.9]	447.5	[442.8, 453.7]	7.9	[0.1, 15.0]	11.4	[5.3, 16.2]
2008Q1	432.3	438.0	[431.6, 445.5]	439.5	[434.0, 445.8]	-5.7	[-13.2, 0.7]	-7.3	[-13.6, -1.7]
2011Q3	431.4	438.4	[432.2, 446.1]	440.8	[435.5, 447.2]	-7.0	[-14.7, -0.7]	-9.4	[-15.8, -4.0]
2015Q1	451.0	444.0	[437.6, 451.4]	445.7	[440.3, 452.2]	7.1	[-0.4, 13.5]	5.3	[-1.1, 10.8]
2020Q2	464.3	447.2	[440.7, 455.2]	448.8	[442.7, 456.2]	17.1	[9.1, 23.6]	15.4	[8.1, 21.6]
2022Q3	469.7	454.8	[447.0, 463.0]	454.3	[447.1, 462.0]	14.9	[6.7, 22.7]	15.4	[7.8, 22.6]
2025Q4	463.4	455.2	[447.1, 463.5]	455.2	[447.6, 463.7]	8.2	[-0.1, 16.3]	8.2	[-0.3, 15.8]

Notes: Estimates of the natural rate of exchange and of dollar misalignment at selected dates. q is 100 log of the BIS real narrow effective exchange rate. One-sided estimates use information through the quarter shown; two-sided estimates use the total sample. Misalignment is $q_t - q_t^*$ in log points, positive when the dollar is above its natural rate. Bands are 90 percent and combine filter with parameter uncertainty, conditional on the imposed signal-to-noise ratios λ_g and λ_q of trend growth and the natural rate respectively.

Table 2: Parameter estimates

		Estimate	(s.e.)
<i>IS curve</i>			
a_1	Output gap, first lag	1.212	(0.098)
a_2	Output gap, second lag	-0.257	(0.100)
a_q	Real exchange rate gap	0.010	(0.006)
a_r	Real rate gap	-0.068	(0.022)
σ_y	Disturbance s.d.	0.644	(0.061)
<i>Phillips curve</i>			
b_π	Weight on first lag of inflation	0.706	(0.065)
b_y	Output gap	0.049	(0.027)
b_q	Real exchange rate gap	-0.010	(0.007)
σ_π	Disturbance s.d.	0.863	(0.041)
<i>Real exchange rate</i>			
$\rho_{q,1}$	Own lag of the gap, first	1.203	(0.076)
$\rho_{q,2}$	Own lag of the gap, second	-0.293	(0.070)
c_y	Output gap	-0.008	(0.080)
c_r	Interest differential	0.654	(0.226)
σ_q	Disturbance s.d.	2.070	(0.100)
<i>Natural rate of exchange</i>			
β_τ	Terms of trade	0.588	(0.110)
σ_μ	Innovation s.d.	0.813	(-)
<i>Net exports</i>			
γ_q	Real exchange rate gap	-0.006	(0.003)
γ_y	Output gap	0.018	(0.010)
γ_τ	Change in the terms of trade	0.081	(0.014)
σ_n	Disturbance s.d.	0.289	(0.014)
<i>Trend output</i>			
σ_{y^*}	Innovation s.d., level	0.304	(0.097)
σ_g	Innovation s.d., growth	0.021	(-)
<i>COVID volatility scale</i>			
κ_0	2020Q2	6.780	(2.478)
κ_1	2020Q3	7.050	(2.655)
ρ_κ	Decay	0.051	(0.141)

Notes: Final stage maximum likelihood estimates from 1970Q1 to 2025Q4. Dashes mark imposed parameters, with σ_μ set through the median-unbiased ratio $\lambda_q = 0.39$ and σ_g through $\lambda_g = 0.0684$. Standard errors are computed from the numerical Hessian of the log-likelihood and mapped to the natural parameters by the delta method. As is usual for this type of procedure, they do not account for the estimation of λ_g and λ_q .

Table 3: Robustness and sensitivity to the trend/cycle ratio

	$\log L$	$\rho_{q,1} + \rho_{q,2}$	σ_μ	σ_{q-q^*}	$q_t - q_t^*$	
					1985Q1	2025Q4
<i>Panel A. Specification</i>						
Baseline	-1071.8	0.910	0.81	9.1	18.0	8.2
2020Q2–Q4 treated as missing	-1034.9	0.891	0.81	8.7	19.5	5.5
No COVID adjustment	-1154.4	0.910	0.83	9.4	17.3	9.7
No terms of trade, $\beta_\tau = 0$	-1084.8	0.945	0.88	10.3	18.7	13.0
No interest differential, $c_r = 0$	-1076.6	0.971	0.84	8.3	12.0	-0.6
Net exports in levels, quadratic trend	-1066.8	0.910	0.81	9.1	17.5	8.0
No gap, $a_q = b_q = \gamma_q = 0$	-1074.4	0.891	0.81	8.9	18.8	6.5
Terms of trade lagged, τ_{t-1}	-1084.8	0.946	0.88	10.4	18.7	13.2
Real differential, four-quarter inflation	-1076.5	0.975	0.84	8.2	11.4	-1.4
Real differential, five-year inflation	-1075.2	0.953	0.83	8.4	17.5	4.9
AR(1) exchange rate gap, $\rho_{q,2} = 0$	-1079.7	0.909	0.84	9.0	17.8	7.6
No $\Delta\tau_t$ in net exports, $\gamma_\tau = 0$	-1088.2	0.911	0.82	9.3	18.1	8.6
Both: the previous specification	-1096.2	0.911	0.85	9.1	17.3	8.7
<i>Panel B. Imposed ratio λ_q</i>						
Estimated jointly	-1071.8	0.910	0.78	9.2	18.1	8.3
$\lambda_q = 0.00$, lower 90% limit	-1072.9	0.930	0.00	9.0	22.1	6.9
$\lambda_q = 0.39^\dagger$	-1071.8	0.910	0.81	9.1	18.0	8.2
$\lambda_q = 3.00$, upper limit (grid edge)	-1075.2	0.932	2.14	6.3	12.6	3.5

Notes: Columns report the log-likelihood, the sum of the autoregressive coefficients on the exchange rate gap, the standard deviation of innovations to the natural rate, the standard deviation of the two-sided gap, and the value of the gap at its peak and at the end of the sample. Panel A varies the specification, holding λ_g and λ_q at their baseline values. Panel B instead varies λ_q , imposing the lower and upper ends of its 90 percent interval, with $\dagger$ marking the baseline estimate. Panel B also reports the case in which λ_q is estimated jointly with the other parameters by maximum likelihood. Each specification is estimated from the baseline estimates as well as from the usual grid of starting values.

The natural rate of exchange for the United States

Two-sided (smoothed) estimate, with 68% and 90% bands

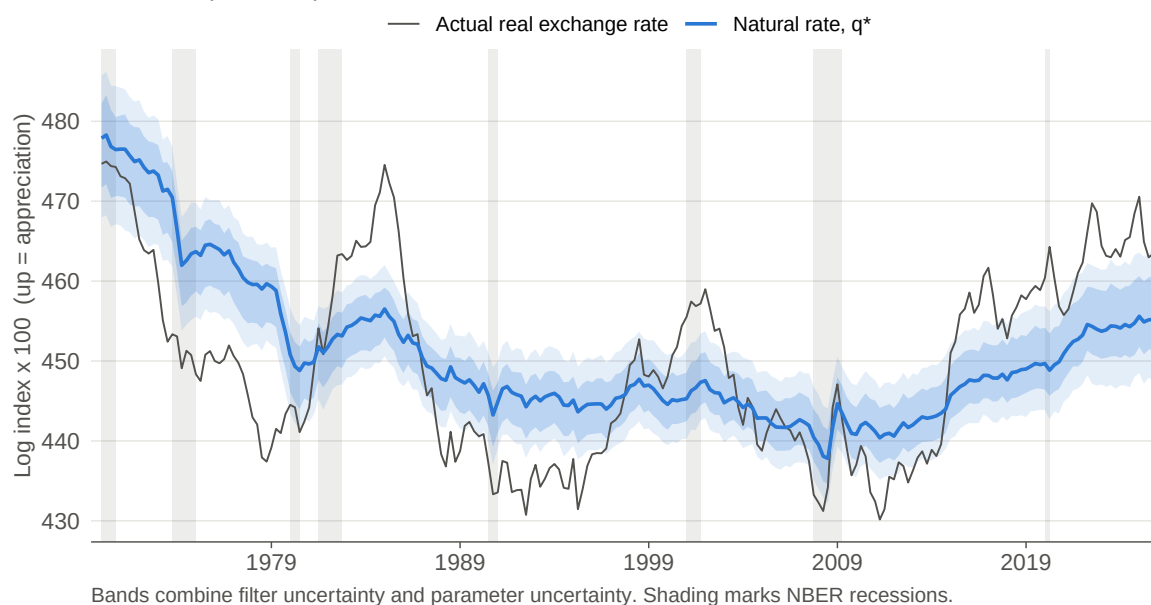


Figure 1: The natural rate of exchange for the United States

Notes: Two-sided estimate of q^* with 68 and 90 percent bands combining filter and parameter uncertainty, against the actual real effective exchange rate. Both time-series are 100 times the log of the index, 2020 = 100. Shading marks recessions dated by the National Bureau of Economic Research.

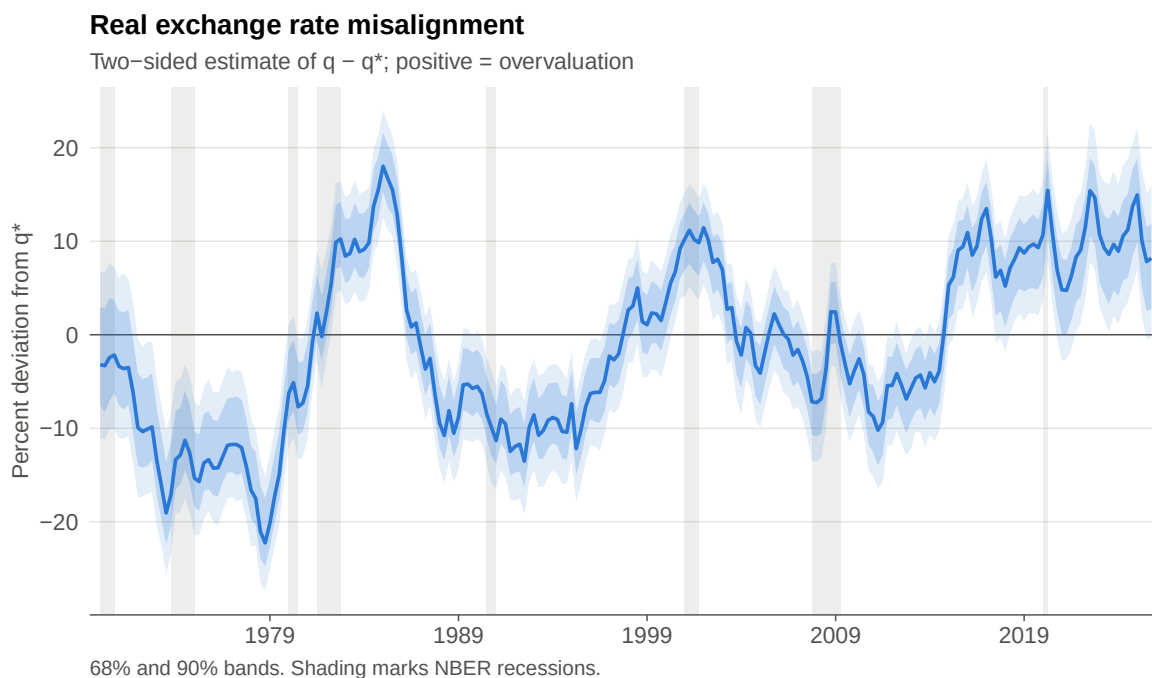


Figure 2: Dollar misalignment

Notes: Smoothed estimate of misalignment, $q_t - q_t^*$, measured as the difference in 100 times the log. Positive values indicate a real exchange rate above its natural rate (overvaluation) and negative values a rate below it (undervaluation). Shaded bands are 68 and 90 percent intervals combining filter and parameter uncertainty.

A Data

The sample is 1970Q1–2025Q4, all series are quarterly. Output is real gross domestic product (FRED `GDPC1`), entered as 100 log. Inflation is the annualised quarterly log change in the core personal consumption expenditures index (`PCEPILFE`), $400 \Delta \log$. The real exchange rate is the Bank for International Settlements real narrow effective exchange rate for the United States (`RNUSBIS`), entered as 100 log and oriented so that an increase is an appreciation. Net exports are nominal net exports of goods and services (`NETEXP`) as a percent of nominal gross domestic product (`GDP`). The terms of trade, τ_t , are 100 log of the ratio of the implicit price deflator for exports (`A020RD3Q086SBEA`) to that for imports (`A021RD3Q086SBEA`). The real rate gap is the federal funds rate, less a four-quarter moving average of core PCE inflation, less the one-sided estimate of the natural rate of interest, all three as published by the Federal Reserve Bank of New York for [Holston et al. \(2023\)](#). The interest differential is the ten-year Treasury yield (`GS10`) less a weighted average of the long-term government bond yields of Canada, France, Germany and the United Kingdom (`IRLTLT01{CA,FR,DE,GB}Q156N`), the G7 economies whose series cover the whole sample; Japan and Italy begin only in 1989 and 1991. The weights are each economy’s share of the four economies’ GDP in current dollars (World Bank), averaged from 1970 and held fixed. These approximate the trade weights of the BIS index only roughly. The real alternatives use OECD consumer prices, which end in 2025Q1, extended with Eurostat harmonised prices for Germany and France and by the previous year’s average growth for Canada and the United Kingdom. Monthly series are averaged to quarterly frequency. As q_t is an index, the level of q_t^* inherits its normalisation (2020 = 100); misalignment $q_t - q_t^*$ is invariant to it and is the object we interpret. Deviations are differences in 100 times the log, which read as approximate percentages when small; where a large change is quoted, the exact percentage is given alongside.

B Model specification

The system is estimated in three stages, in the state-space notation of [Hamilton \(1994\)](#). The observation and state equations are

$$\mathbf{y}_t = \mathbf{A}'\mathbf{x}_t + \mathbf{H}'\boldsymbol{\xi}_t + \mathbf{w}_t, \quad \boldsymbol{\xi}_t = \mathbf{F}\boldsymbol{\xi}_{t-1} + \mathbf{v}_t, \quad \mathbf{w}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_t), \quad \mathbf{v}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_t), \quad (11)$$

where $\mathbf{x}_t$ collects the predetermined and exogenous variables:

$$\begin{aligned} \mathbf{y}_t &= (y_t, \pi_t, q_t, n_t)', \\ \mathbf{x}_t &= (y_t, y_{t-1}, y_{t-2}, \pi_{t-1}, \bar{\pi}_{t-2,t-4}, q_{t-1}, q_{t-2}, \tau_t, \tau_{t-1}, \tau_{t-2}, \tilde{r}_{t-1}, \Delta i_{t-1}, n_{t-1}, \Delta \tau_t)', \\ \boldsymbol{\xi}_t &= (y_t^*, y_{t-1}^*, y_{t-2}^*, g_t, \mu_t, \mu_{t-1}, \mu_{t-2})'. \end{aligned}$$

Current output y_t enters $\mathbf{x}_t$ only through the net-exports equation, whose output gap is contemporaneous; the system is recursive in it, so the likelihood is unaffected. Because every gap is an observable less a state, $\tilde{y}_t = y_t - y_t^*$ and $\tilde{q}_t = q_t - \beta_\tau \tau_t - \mu_t$, the loadings are

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_1 & -a_2 & 0 & 0 & -a_q & 0 \\ 0 & -b_y & 0 & 0 & 0 & -b_q & 0 \\ 0 & -c_y & 0 & 0 & 1 & -\rho_{q,1} & -\rho_{q,2} \\ -\gamma_y & 0 & 0 & 0 & 0 & 0 & -\gamma_q \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

and the rows of $\mathbf{A}'$ collect the corresponding observable terms, products of parameters included. The covariance matrices are diagonal, with

$$\begin{aligned} \text{diag}(\mathbf{Q}) &= (\sigma_{y^*}^2, 0, 0, \sigma_g^2, \sigma_\mu^2, 0, 0), \\ \text{diag}(\mathbf{R}_t) &= \kappa_t^2(\sigma_y^2, \sigma_\pi^2, \sigma_q^2, \sigma_n^2), \end{aligned}$$

where κ_t is the volatility scale of [Section 2.4](#), equal to one before 2020Q2. Every measurement equation carries its own disturbance, so $\mathbf{R}_t$ is non-singular; the zeros in $\text{diag}(\mathbf{Q})$ correspond to the lag-shift states, which carry no independent disturbance.

B.1 Multi-stage maximum likelihood

Stage I. Imposes $\sigma_g = 0$, so that trend growth is constant and g is a parameter rather than a state. The vector estimated by maximum likelihood is

$$\theta_I = (a_1, a_2, a_q, a_r, b_\pi, b_y, b_q, \rho_{q,1}, \rho_{q,2}, c_y, c_r, \beta_\tau, \gamma_q, \gamma_y, \gamma_\tau, \\ g, \sigma_y, \sigma_\pi, \sigma_q, \sigma_n, \sigma_{y^*}, \sigma_\mu, \kappa_0, \kappa_1, \rho_\kappa).$$

The ratio $\lambda_g = \sigma_g/\sigma_{y^*}$ follows from the [Stock and Watson \(1998\)](#) median-unbiased estimator applied to the structural-break statistic of the smoothed potential-output growth path $\Delta\hat{y}_{t|T}^*$.

Stage II. Imposes $\sigma_g = \hat{\lambda}_g\sigma_{y^*}$, so that g becomes a state, and holds the latent component of the natural rate constant, $\sigma_\mu = 0$. The remaining parameters other than $\rho_{q,1}$ and $\rho_{q,2}$ are estimated by maximum likelihood. With μ constant, (4) implies that

$$z_t \equiv \varphi_t - \rho_{q,1}\varphi_{t-1} - \rho_{q,2}\varphi_{t-2} - c_y\tilde{y}_{t-1} - c_r\Delta i_{t-1} = (1 - \rho_{q,1} - \rho_{q,2})\mu + \varepsilon_t^q, \quad \varphi_t \equiv q_t - \beta_\tau\tau_t,$$

has a constant mean. The exponential Wald statistic for a shift in that mean at an unknown date gives $\lambda_q = \sigma_\mu/\sigma_q$ through the median-unbiased estimator, with the mapping from the ratio to the statistic simulated for the exchange rate equation. Because that mapping depends on $(\rho_{q,1}, \rho_{q,2})$, and their Stage III estimates depend on λ_q , they are held at their Stage III values and the two stages are iterated to a fixed point.

Stage III. Imposes both ratios, so that the fourth and fifth elements of $\text{diag}(\mathbf{Q})$ are $\hat{\lambda}_g^2\sigma_{y^*}^2$ and $\hat{\lambda}_q^2\sigma_q^2$. The vector estimated by maximum likelihood is $\theta_{III} = \theta_I \setminus \{g, \sigma_\mu\}$, of dimension 23. One- and two-sided state estimates follow from the filter and the smoother at $\hat{\theta}_{III}$, and the reported intervals additionally integrate over its asymptotic distribution.

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